Non-Gaussianity bounded uncertainty relation for mixed states

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We derive new uncertainty relation for one-dimensional mixed state given its purity and degree of non-Gaussianity. This relation extends the purity-bounded uncertainty relation for mixed states derived by V. V. Dodonov and V. I. Man'ko. For the special case of pure states it provides us with an extended version of the Robertson-Schrödinger uncertainty relation, saturated by a set of states which includes all the eigenstates of the quantum harmonic oscillator. We represent our results as a bound in a three-dimensional parametric space of mixed states and identify the regions of the bound realized by states with non-negative Wigner function. This takes us closer to a proper extension of Hudson's theorem to mixed states and permits us to visualize and compare the set of states with non-negative Wigner function and the set of states which minimize the newly derived uncertainty relation.

I. INTRODUCTION

The only pure states which saturate the Schrödinger-Robertson [1] uncertainty relation for the canonically conjugated coordinate and momenta, are the Gaussian states. The latter, due to Hudson's theorem [2], are also the only pure states with strictly positive Wigner function [3]. One may naturally wonder how these theorems can be extended to mixed states, and whether the "Gaussian" link between them persists.

In the space of mixed states, different forms of uncertainty relation have been suggested varying with the choice of the quantities characterizing the mixed states i.e. purity, purities of higher order or various entropies (see [4] for a review). Among these proposals, a basic one, i.e. expressed in a closed form, is the *purity-bounded uncertainty relation* suggested by Dodonov and Man'ko [5]. According to this relation the minimum uncertainty is a function of the purity characterizing the degree of the mixedeness of a state. In a more recent work Dodonov [4] has proven that the mixed states which minimize this relation possess positive Wigner functions and that they are not extremely "far" from Gaussian states of the same purity. If instead of the purity, the von Neumann entropy is employed as a measure degree of mixedeness, the results are not identical. Bastiaans [6] has derived an uncertainty relation based on the von Neumann entropy of mixed states, the *entropy-bounded uncertainty relation* and has proven that the mixed states of minimum uncertainty are the thermal (Gaussian) states.

These results already give a preliminary answer to the questions we posed suggesting that, the link between the ability to minimize relevant uncertainty relations and positivity of Wigner functions seem to persist in the space of mixed states, though the Gaussianity is no longer a necessary requirement for extremality of the uncertainty. However, to be able to answer our questions in more complete way we need to have a closer view on the set of mixed states with positive Wigner functions.

In a recent work [7] we have formulated an extension of Hudson's theorem, by identifying the maximum degree of deviation from a Gaussian state which can be reached by a mixed state with positive Wigner function, given its purity and its uncertainty. This problem does not have a simple solution and in [7] we have only identified the bounds for any continuous classical distribution which may not necessarily correspond to a valid Wigner function, and therefore the bounds are not tight for the set of quantum states.

In view of this discussion, we extend in this work the purity-bounded uncertainty relation by adding one more parameter, the non-Gaussianity characterizing the distance of the state from a Gaussian state with the same covariance matrix. In this way we are able to draw more complete conclusions on the overlap between states minimizing this uncertainty relation and the set of states with positive Wigner function. In addition, this extension permits us to show explicitly how the Gaussian states smoothly become extremal pure states in the context of uncertainty and positivity of Wigner's functions.

In Sec. II, we introduce the quantities which we choose to characterize one-dimensional mixed states and shortly review the already acquired results concerning uncertainty relations. In Sec. III we derive new bound on the uncertainty of a mixed state characterized by its purity and non-Gaussianity. We name this relation the *non-Gaussianity bounded uncertainty relation.* In Sec. IV we identify numerically the parts of the bound realized by states with positive Wigner function, thus visualizing the overlap of the set of states with positive Wigner function and the states minimizing the non-Gaussianity bounded uncertainty relation.

In a recent work [8] we have extended the entropy-bounded uncertainty relation using similar methods as we use here. However, that extension, as well as, the entropy-bounded uncertainty relation itself requires numerical methods to be expressed in a proper form. In Sec.V, we discuss the main outcomes of this work and we compare them with those in [8].

II. OUR PARAMETRIC SPACE

For our purposes we need first to choose a convenient parametrization of the space of mixed states that will permit us to formulate the extensions of the uncertainty relation and Hudson's theorem in a unified manner.

Let us start with a general density matrix $\hat{\rho}$ determining a quantum state. Apparently a measure of the degree of mixedeness of the state is an indispensable parameter characterizing a mixed state. We choose as the most convenient measure for us, the *purity* $\mu = \mathrm{Tr}\left(\hat{\rho}^2\right)$.

The second basic characteristic of the state $\hat{\rho}$ is its *covariance matrix* γ defined as

$$\gamma_{ij} = \mathrm{Tr}(\{(\hat{r}_i - d_i), (\hat{r}_j - d_j)\}\hat{\rho}) \tag{1}$$

where $\hat{\mathbf{r}}$ is the vector of the canonically conjugated observables position and momentum (or quadrature operators in quantum optics) $\hat{\mathbf{r}} = (\hat{x}, \hat{p})^T$, $\mathbf{d} = Tr(\hat{\mathbf{r}}\hat{\rho})$ is the displacement vector, and $\{\cdot, \cdot\}$ is the anticommutator. We can put the displacement vector to zero with no loss of generality since the purity (as well as the quantities we will be interested in) is invariant with respect to $\mathbf{d}$.

The importance of the covariance matrix stems from the fact that it is measurable experimentally and that it is tightly connected, as we show below, to the uncertainty relation. Throughout the text we will consider quantum systems with only one pair of conjugate variables, like the system of one particle or one-mode state in quantum optics, therefore we consider 2×2 covariance matrices. The covariance matrix of $\hat{\rho}$ uniquely determines a Gaussian state $\hat{\rho}_G$ which henceforth we will refer to as the *reference Gaussian state* for the state $\hat{\rho}$ and we choose as second parameter the purity of $\hat{\rho}_G$,

$$\mu_G = 1/\sqrt{\gamma_{11}\gamma_{22} - |\gamma_{12}|^2}. \tag{2}$$

As a third parameter we introduce a measure of the *non-Gaussianity*, i.e., the distance of the state from its reference Gaussian state. At the first step we are going to work with the trace overlap $\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right)$ between the states $\hat{\rho}$ and $\hat{\rho}_G$. Although the trace overlap is not a measure of distance, it can be employed together with μ_G, μ for evaluating a correct measure of non-Gaussianity, the normalized Hilbert-Schmidt distance [4],[9],

$$\delta = \frac{1}{2\mu}\mathrm{Tr}\left((\hat{\rho} - \hat{\rho}_G)^2\right) = \frac{\mu_G + \mu - 2\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right)}{2\mu}. \tag{3}$$

Therefore, while a state $\hat{\rho}$ requires an infinite set of parameters to be fully described, it is enough for our purposes to work in the reduced parametric space $\{\mu, \mu_G, \delta\}$. In [7] we have used the same parametric space. There we have formulated an extension of Hudson's theorem to mixed states by identifying an upper bound, though not tight, on the non-Gaussianity δ for states with positive Wigner function. This bound is represented by a surface in the space $\{\mu, \mu_G, \delta\}$.

Let us we note that the existing purity-bounded uncertainty relation [5] can be represented as a line-bound in the plane $\{\mu, \mu_G\}$. Before we justify this statement, we shortly review the historical evolution of the knowledge concerning uncertainty relations and for more details we address an interested reader to [5]. For pure states of covariance matrix γ, the first uncertainty relation

$$\gamma_{11}\gamma_{22} \approx 4 \tag{4}$$

(where $\hbar = 1$) was proposed by Heisenberg [10] and proven in its more strict form, $\gamma_{11}\gamma_{22} \geqslant 1$, by Kennard [11] in 1927 . This uncertainty relation is saturated by coherent states and squeezed states whose major and minor axis in the phase-space, are parallel to the x- and p-axis. Few years later, the Heisenberg uncertainty relation was generalized by Schrödinger and Robertson, independently, to a relation valid for any set of non-commuting observables and which, in the case of $(\hat{x}, \hat{p})$, takes the form

$$\sqrt{\gamma_{11}\gamma_{22} - |\gamma_{12}|^2} \geq 1. \tag{5}$$

The Scrödinger-Robertson uncertainty relation accounts for the possible correlations between the coordinates and therefore is saturated by all Gaussian pure states. Even though, the inequality Eq. (5) is valid also for mixed states it cannot be saturated by any mixed state. Therefore, for the sets of states with purity less than one, this uncertainty relation is not tight. A tight uncertainty relation for mixed states is the purity-bounded uncertainty relation derived by Dodonov and Man'ko [5]

$$\sqrt{\gamma_{11}\gamma_{22} - |\gamma_{12}|^2} \geq \Phi\left(\mu\right). \tag{6}$$

For mixed states the lower limit on the uncertainties depends on the purity of the state via a monotonic function Φ satisfying

$$\Phi\left(1\right) = 1 \text{ and } 1 \leq \Phi\left(\mu\right) \leq 1/\mu. \tag{7}$$

In view of Eq. (2) it is then evident that the purity-bounded uncertainty relation Eq. (6) is equivalent to a bound given by a line in the $\{\mu, \mu_G\}$ plane. Dodonov and Man'ko have proven [5] that this bound is realized by states with phase-independent and positive Wigner functions and they have derived an exact but piecewise expression for the function Φ in Eq.(6). In A we present an alternative, though parametric, expression for this line-bound whose construction permits for a generalization to the three dimensional parametric space. We also note here a quite unexpected fact visualized by the purity-bounded uncertainty relation. It is known that on one hand the pure states which saturate the Scrödinger-Robertson uncertainty relation for pure states are the Gaussian states. On the other hand the Gaussian mixed states (thermal states) maximize the von Neumann entropy [6] being, in a sense, maximally "disordered" [12]

given μ_G Eq.(2). However, contrary to our intuition, the purity-bounded uncertainty relation dictates that the mixed Gaussian states are not the states which maximize the purity.

In the next section, we extend the purity-bounded uncertainty relation by including into it the non-Gaussianity δ of states

$$\sqrt{\gamma_{11}\gamma_{22}-|\gamma_{12}|^2} \geq F(\mu,\delta). \qquad (8)$$

For our derivation and in view of Eq. (2) it is convenient (as we explain in more detail in Sec.III) to conceive the inequality given by Eq.(8) as a bound in the introduced parametric space $\{\mu,\mu_G,\delta\}$. This bound valid for all mixed states is then compared, in Sec.IV, with the bound for states with positive Wigner functions.

III. NON-GAUSSIANITY BOUNDED UNCERTAINTY RELATION

In order to find function F in Eq.(8) it is enough to find the bound for the whole set of mixed quantum states in the parametric space $\{\mu,\mu_G,\delta\}$. It is obvious that such a bound will be a surface (since the space is three-dimensional) and if it is single-valued in the parameters μ and δ it would represent the function $F(\mu,\delta)$. Our results show that the bound is in general single-valued, however there is some particular domain of $\{\mu,\delta\}$. where the bound is double-valued. This problem does not emerge if one works with $\mathrm{Tr}(\hat{\rho}\hat{\rho}_G)$ instead of δ (see Eq. (3)). That is why we derive our results employing the trace overlap. With the help of Eq. (3) the results can be directly translated to the initial parametric space $\{\mu,\mu_G,\delta\}$ and we do so, mainly for the visualization of the results.

A possible way to proceed with the derivation of the bound is to search for the states which minimize or maximize the purity μ if the values of the other two parameters, μ_G and $\mathrm{Tr}(\hat{\rho}\hat{\rho}_G)$, are kept constant. We conceive then the total bound consisting of two regions, namely, the *region I* where the purity μ is minimized and the *region II* where the purity is maximized given μ_G and $\mathrm{Tr}(\hat{\rho}\hat{\rho}_G)$. At the final step we recombine both regions and present the whole bound which we name *non-Gaussianity bounded uncertainty relation* as the abbreviation of the more exact title *non-Gaussianity and purity bounded uncertainty relation.*

A. *Region I*: States of minimum purity

Let us consider a general density matrix $\hat{\rho}$ characterized by the purity μ_G of the reference Gaussian state $\hat{\rho}_G$ and the trace overlap between $\hat{\rho}$ and $\hat{\rho}_G$, $\mathrm{Tr}(\hat{\rho}\hat{\rho}_G)$. By using symplectic transformations we can set the covariance matrix γ of $\hat{\rho}$ into a symmetric form where $\gamma_{11}=\gamma_{22}$ and $\gamma_{12}=0$. Symplectic transformations, such as squeezing and rotation, do not change the quantities of interest (see ref.[7]) and therefore without loss of generality, we can assume that the reference Gaussian state for $\hat{\rho}$, $\hat{\rho}_G$, is a thermal state. In the Wigner representation such a state, $\hat{\rho}_G$, is

$$W_G(r)=\frac{\mu_G}{\pi}e^{-r^2\mu_G}, \qquad r=\sqrt{x^2+p^2} \qquad (9)$$

where μ_G is its purity defined in Eq. (2).

The next step is to prove the *Lemma* stating that for a given state $\hat{\rho}$ with these characteristics, there is always a state of the same characteristics possessing a phase-independent Wigner function, that is of equal or lower purity than $\hat{\rho}$. To prove the Lemma it is more convenient to work in the Wigner phase-space representation. In the general case, quantum state $\hat{\rho}$ possesses an angular-dependent Wigner function $W(r,\varphi)$ and its purity can be expressed as

$$\mu=2\pi\iint W(r,\varphi)^2\, r\mathrm{d}r\mathrm{d}\varphi. \qquad (10)$$

The trace overlap between $\hat{\rho}$ and $\hat{\rho}_G$ is

$$\mathrm{Tr}(\hat{\rho}\hat{\rho}_G)=2\pi\iint W(r,\varphi)\,W_G(r)\, r\mathrm{d}r\mathrm{d}\varphi. \qquad (11)$$

Now let us construct a new state $\hat{\rho}_s$ with phase-invariant Wigner function $W_s(r)$, by phase-averaging the Wigner function $W(r,\varphi)$,

$$W_s(r)=\frac{1}{2\pi}\int W(r,\varphi)\,d\varphi. \qquad (12)$$

The reference Gaussian state for $\hat{\rho}_s$ will be the same as for $\hat{\rho}$, since phase-averaging cannot affect the angular-independent Wigner function Eq.(9). The trace overlap among the symmetrized state $\hat{\rho}_s$ and $\hat{\rho}_G$ will remain the same, as well. This is a straightforward result of substitution of the phase-independent Wigner function given by Eq. (12) into Eq. (11). However, the purity μ_s of the symmetrized state $\hat{\rho}_s$ is constrained to be smaller than, or equal to, that of $\hat{\rho}$. Indeed, by applying the Cauchy-Schwarz inequality we have

$$\begin{aligned}\mu_s &= 2\pi\iint W_s(r)^2\, r\mathrm{d}r\mathrm{d}\varphi \\ &= \iiint W(r,\varphi)\,W(r,\Phi)\, r\mathrm{d}rd\varphi d\Phi \\ &\leq \sqrt{\iiint W(r,\varphi)^2\, r\mathrm{d}rd\varphi d\Phi}\sqrt{\iiint W(r,\Phi)^2\, r\mathrm{d}rd\varphi d\Phi} \\ &= 2\pi\iint W(r,\varphi)^2\, r\mathrm{d}r\mathrm{d}\varphi=\mu.\end{aligned}$$

This concludes the proof of our Lemma.

From this Lemma is straightforward to deduce that the bound corresponding to the states of minimum purity is achieved by states with angular-independent Wigner function and we proceed by applying the method of Lagrange multipliers to identify these states. We note also

here that this Lemma is well in accordance with the result of Dodonov and Man'ko [5] who have shown that for any given μ_G one can always choose a coordinate system where the states with angular-independent Wigner functions minimize the purity μ.

Any state possessing an angular-independent Wigner function can be expressed [13] as a convex combination of the *number* (Fock) states

$$\hat{\rho} = \sum_{n=0}^{\infty} \rho_n \left|n\right\rangle \left\langle n\right|. \tag{13}$$

In order to identify the weights ρ_n of the states which minimize the purity

$$\mu = \sum_{n=0}^{\infty} \rho_n^2 \tag{14}$$

we apply the method of Lagrange multipliers with the following constraints imposed on the solution,

1. Normalization

$$\sum_{n=0}^{\infty} \rho_n = 1. \tag{15}$$

2. Fixed purity for the reference Gaussian state $\hat{\rho}_G$ (an un-shifted thermal state as in Eq.(9))

$$\sum_{n=0}^{\infty} \rho_n \left(2n+1\right) = 1/\mu_G. \tag{16}$$

3. Fixed overlap between $\hat{\rho}$ and $\hat{\rho}_G$

$$\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right) = \sum_{n=0}^{\infty} \rho_n \frac{2\mu_G \left(1-\mu_G\right)^n}{\left(1+\mu_G\right)^{n+1}}. \tag{17}$$

Applying the Lagrange multipliers method on Eq.(14) with the constrains Eqs.(15)-(17), we obtain the weights which realize the extremum solution

$$\rho_n = A_1 + A_2 n + A_3 \frac{\mu_G \left(1-\mu_G\right)^n}{\left(1+\mu_G\right)^{n+1}} \tag{18}$$

with the coefficients A_i determined by the conditions $1-3$.

In order to restrict ourselves to positive density matrices we replace the index n in Eq.(18) by a continuous variable x and introducing instead of ρ_n the function

$$f\left(x\right) = A_1 + A_2 x + A_3 \frac{\mu_G \left(1-\mu_G\right)^x}{\left(1+\mu_G\right)^{x+1}}. \tag{19}$$

The same ansatz is employed in Appendix A to derive an alternative expression for the purity-bounded uncertainty relation Eqs.(62)-(63). One can see that the solution Eq. (18), corresponds to a valid density matrix if and only if $f\left(x\right)$ is convex and positive function in an interval $\left[x_1, x_2\right]$ of the positive semi-axis. When this condition is satisfied, the integer parts of the roots x_1, x_2 of the equation $f\left(x\right) = 0$ define respectively the lower and upper limits of summation in Eq. (13). In the case that the equation has only one positive root we denote this root as x_2 and put x_1 equal to zero. With this ansatz, the density operator that maximizes μ in Eq. (14) under the constraints Eqs. (15)-(17), can be re-expressed as

$$\hat{\rho} = \sum_{n=n_{\min}}^{n_{\max}} \left(A_1 + A_2 n + A_3 \frac{\mu_G \left(1-\mu_G\right)^n}{\left(1+\mu_G\right)^{n+1}}\right) \left|n\right\rangle \left\langle n\right| \tag{20}$$

with $n_{\min} = \lceil x_1 \rceil$, $n_{\max} = \lfloor x_2 \rfloor$. (where $\lceil x \rceil$ gives the smallest integer that is greater or equal to x, and $\lfloor x \rfloor$ gives the integer part of x). The rank of the solution $\mathrm{rank}\left(\hat{\rho}\right) = n_{\max} - n_{\min}$ .This ansatz can alternatively be interpreted as adding two unknowns, $n_{\min}$ and $n_{\max}$, into the initial extremization problem and setting two more constrains, $f(x_1) = 0$ and $f(x_2) = 0$, with $f(x)$ given by Eq.(19), but for brevity reasons we do not repeat the procedure.

Now, we are ready to determine the coefficients A_i. For every positive integer value of $n_{\min}$ (including zero) we express the unknown Lagrange multipliers A_1, A_2 and A_3, in terms of μ_G and x_2. More precisely we employ the conditions Eqs. (15)-(16), and (19) which look now as

$$\sum_{n=n_{\min}}^{\lfloor x_2 \rfloor} \left(A_1 + A_2 n + A_3 \frac{\mu_G \left(1-\mu_G\right)^n}{\left(1+\mu_G\right)^{n+1}}\right) = 1 \tag{21}$$

$$\sum_{n=n_{\min}}^{\lfloor x_2 \rfloor} \left(A_1 + A_2 n + A_3 \frac{\mu_G \left(1-\mu_G\right)^n}{\left(1+\mu_G\right)^{n+1}}\right) \left(2n+1\right) = 1/\mu_G \tag{22}$$

$$A_1 + A_2 x_2 + A_3 \frac{\mu_G \left(1-\mu_G\right)^{x_2}}{\left(1+\mu_G\right)^{x_2+1}} = 0. \tag{23}$$

This system of equations in a more explicit form (after the summation over n) is presented in Appendix B. For each non-negative integer $n_{\min}$ one has to solve this linear system of equations. After this, the last step is to substitute the derived expressions for A_i into Eqs.(14) and (17) and obtain parametric expressions for extremal μ^{ex} and $\mathrm{Tr}(\hat{\rho}_G \hat{\rho})^{ex}$ via μ_G and x_2. In order to assure that $n_{\min}$ is the same in all three equations, one has to require that the constrains $f\left(n_{\min}\right) > 0$ and $f\left(n_{\min} - 1\right) < 0$ are satisfied.

The exact solution presented in the Appendix B is rather cumbersome. However, one can derive an approximate solution simply by setting $\lfloor x_2 \rfloor = x_2$. It turns out that for $n_{\min} = 0$ one arrives to the following parametric expression

$$\mu^{ex} = \frac{\mu_G\left(8\left(2x_2+1\right)\mu_G(\mu_G+1)y^{x_2}\left(2x_2\mu_G+\mu_G-3\right)+y^{2x_2}-\left(2x_2+1\right)^2\mu_G{}^4+4\left(4x_2^3+6x_2^2-1\right)\mu_G{}^3\right)}{\left(y^{x_2}\left(\left(2x_2^2+2x_2-1\right)\mu_G{}^2+\left(4x_2+2\right)\mu_G+3\right)+\left(\mu_G+1\right)\left(2x_2\mu_G+\mu_G-3\right)\right)^2}$$
$$+\frac{\mu_G\left(18\left(2x_2^2+2x_2+1\right)\mu_G{}^2+12\left(2x_2+1\right)\mu_G-9+(\mu_G+1)^2\left(2x_2\mu_G+\mu_G-3\right)^2\right)}{\left(y^{x_2}\left(\left(2x_2^2+2x_2-1\right)\mu_G{}^2+\left(4x_2+2\right)\mu_G+3\right)+\left(\mu_G+1\right)\left(2x_2\mu_G+\mu_G-3\right)\right)^2} \tag{24}$$

$$Tr(\rho\rho_G)^{ex} = -\frac{\mu_G\left(-4\left(2x_2+1\right)\mu_G y^{x_2}+(\mu_G-1)y^{2x_2}\left(2x_2\mu_G+\mu_G+3\right)-(\mu_G+1)\left(2x_2\mu_G+\mu_G-3\right)\right)}{y^{x_2}\left(\left(2x_2^2+2x_2-1\right)\mu_G{}^2+\left(4x_2+2\right)\mu_G+3\right)+\left(\mu_G+1\right)\left(2x_2\mu_G+\mu_G-3\right)} \tag{25}$$

where $y=(1+\mu_G)/(1-\mu_G)$, $x_2\geq 2$. These equations represent the global solution in very good approximation.

In Fig. 1 we present the derived solution which we call *region I* of the bound, projected on the planes, *(a)* $\{\mu_G, \mathrm{Tr}\,(\hat\rho\hat\rho_G)\}$, and *(b)* $\{\mu_G,\mu\}$. In Fig. 1(a), the whole bound is confined between the dotted straight line which corresponds to the Gaussian states and the solid line which represents the solution of *rank* 2 realized by states that are mixtures of any two successive number states

$$\hat\rho_{r2} = a\left|n\right\rangle\left\langle n\right| + (1-a)\left|n+1\right\rangle\left\langle n+1\right|. \tag{26}$$

In Fig.1(b), the lowest dashed line represents the purity-bounded [5] uncertainty relation. One can see it is the lower limit of the more general bound derived here. The upper limit of the so far derived bound (the upper solid line) is the *rank* 2 solution Eq.(26). This curve reaches the line $\mu=1$ of pure states (not shown in the graph) only at the points realized by the number states (indicated on the graph as $n=1,2,\ldots$). The area between this line and the $\mu=1$ line remains *not covered* by the derived bound.

In the same figure (Fig.1(b)) one can notice that solid lines form loops which are arranged in horizontal rows. These lines define the borders between different ranks $\mathrm{rank}\,(\hat\rho)=n_{\max}-n_{\min}$ of the solution in Eq.(20). Each horizontal row corresponds to a specific rank indicated in the figure. This classification of the solution according to the ranks is comparable with the results presented in [4]. One can see also that the loops form "columns". Each column corresponds to different value of $n_{\min}$ which is equal to the number n of the number state indicated at the upper limit of the right border of the column.

B. Region II: States of maximum purity

In Fig. 1(b) we have observed that the states which minimize the purity do not realize the whole wanted bound, meaning that *not* for all pairs of values of $\{\mu_G,\mu\}$ we have obtained a bound on $\mathrm{Tr}\,(\hat\rho\hat\rho_G)$. This fact is not surprising since the trial solution, composed by mixtures of number states, can achieve purity $\mu=1$, only for the number states. The rest of the bound which we call *region II,* must be realized by states which maximize the purity, if μ_G and $\mathrm{Tr}\,(\hat\rho\hat\rho_G)$ are given.

In order to obtain these states we apply first the Lagrange multipliers method for a *generic* density matrix $\hat\rho$. Thus we obtain the extremum, i.e. minimum or maximum value of the purity for a state $\hat\rho$ with given μ_G and $\mathrm{Tr}\,(\hat\rho\hat\rho_G)$. In addition we require, without loss of generality, that covariance matrix of $\hat\rho$ is symmetric in x and p. We express these constrains on the solution in the following way,

1. The state is normalized $\mathrm{Tr}\,(\hat\rho)=1$.

2. The reference Gaussian state is a thermal non-displaced one,

$$\mathrm{Tr}\,(\hat\rho\hat x)=\mathrm{Tr}\,(\hat\rho\hat p)=\mathrm{Tr}\,(\hat\rho\hat x\hat p)=0 \tag{27}$$

and its purity is fixed

$$\mathrm{Tr}\,(\hat\rho\,(2\hat n+1))=\frac{1}{\mu_G}. \tag{28}$$

3. Fixed overlap with $\hat\rho_G$

$$\mathrm{Tr}\,(\hat\rho\hat\rho_G)=\mathrm{Tr}\left(\hat\rho\frac{e^{\beta\hat n}}{N}\right)$$

where $\mathrm{e}^{\beta}=\frac{1-\mu_G}{1+\mu_G}$ and $N=\frac{2\mu_G}{1+\mu_G}$ the normalization factor.

The solution to this extremization problem is of the form

$$\hat\rho_{ex}=\alpha_0+\alpha_1\hat x+\alpha_2\hat p+\alpha_3\hat x\hat p+\alpha_4\hat n+\alpha_5 e^{\beta\hat n}. \tag{29}$$

However the conditions given by Eq. (27) dictate that $\alpha_1=\alpha_2=\alpha_3=0$ and therefore that the solution must be diagonal in the eigenbasis of the harmonic oscillator. In other words, the extremization problem on a generic density matrix is reduced to the problem we have solved in the Sec. IIIA and it provides the *region I* of the bound. Therefore, the method of Lagrange multipliers does not add anything new and in order to find the states in the *region II* maximizing purity we have to consider a possibility of the existence of a degenerate (invariant) subspace of solutions.

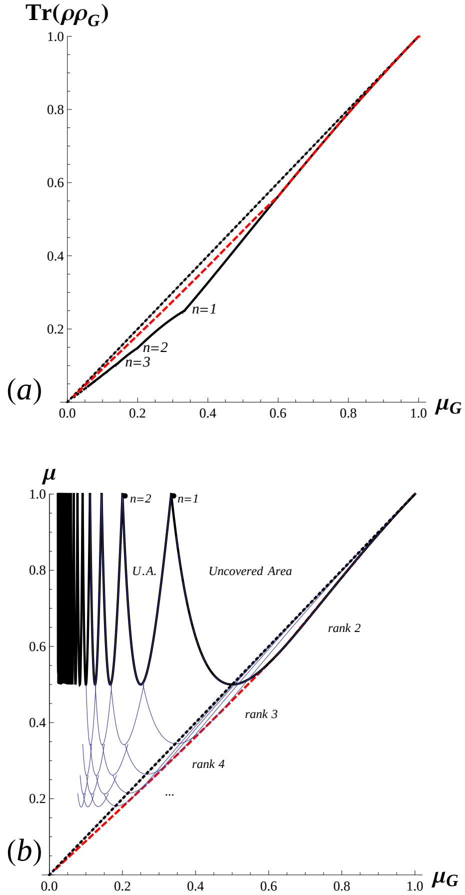

FIG. 1: The *region I* of the bound projected *(a)* on the $\{\mu_G, \mathrm{Tr}\,(\rho\rho_G)\}$ plane and *(b)* on the $\{\mu, \mu_G\}$ plane. The dotted (straight) line stands for the Gaussian states, the bold line for the solution of rank 2 Eq.(26) and the dashed line for the purity-bounded uncertainty relation. In part *(b),* the solid lines split the bound in vertical segments according to different $n_{\min}$ and in horzontal segments according to the rank $n_{\max} - n_{\min}$ of the density matrices of the solution Eq.(20). In the figure only the ranks $2-6$ and $n_{\min}:0-3$ are depicted.

1. *Pure states*

Let us now proceed in a different way and try to process the extremization problem in a modified way to reveal the degeneracy. Instead of working with the mixing coefficients of the density matrix, we consider them fixed and we make variation on the wavevectors, a mixed state can be decomposed on.

To clarify the procedure we start with the special case of pure states where $\hat{\rho} = |\psi\rangle\langle\psi|$ and $|\psi\rangle = \sum_n \psi_n |n\rangle$ in an eigenbasis which for convenience we choose to be the one of the quantum harmonic oscillator. If we restrict ourselves to states which possess as a non-displaced reference Gaussian state then the non-vanishing amplitudes ψ_n in the superposition should be at least separated by three vanishing ones, i.e. $|\psi\rangle = \sum_m \psi_m |i_m\rangle$ where $i_{m+1} - i_m \geq 3$. Since, the purity is fixed to 1 we have to extremize another quantity which we choose to be the overlap

$$\begin{aligned} \mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G) &= \langle\psi| e^{\beta\hat{n}} |\psi\rangle / N \\ &= \frac{2\mu_G}{1+\mu_G} \sum_n |\psi_n|^2 \left(\frac{1-\mu_G}{1+\mu_G}\right)^n \end{aligned} \tag{30}$$

assuming in addition that the pure states are normalized $\langle\psi|\,\psi\rangle = 1$ and of fixed covariance matrix

$$\begin{aligned} 1/\mu_G &= \langle\psi| (2\hat{n}+1) |\psi\rangle \\ &= \sum_n |\psi_n|^2 (2n+1). \end{aligned} \tag{31}$$

The functional to variate is now the following

$$\begin{aligned} f(\{\psi_i\}) &= \langle\psi| \frac{e^{\beta\hat{n}}}{N} |\psi\rangle \\ &+ a_1 \langle\psi|\,\psi\rangle + a_2 \langle\psi| (2\hat{n}+1) |\psi\rangle\,. \end{aligned} \tag{32}$$

After differentiating over the amplitudes ψ_n which without loss of generality we can consider as real numbers, we derive the following condition on the solution $|\psi\rangle$ to the extremization problem

$$\left(e^{\beta\hat{n}} + c_1 + c_2\hat{n}\right)|\psi\rangle = 0. \tag{33}$$

In this form one can see that apart from the obvious solution where $|\psi\rangle = |n\rangle$ there is one more possibility that was not revealed when we applied the extremization procedure on the density matrices, i.e., a superposition of two number states

$$|\psi\rangle = \psi_n |n\rangle + \psi_{n+i} |n+i\rangle \tag{34}$$

with $i \geqslant 3$ (so that the covariance matrix is symmetric) and $|\psi_n|^2 + |\psi_{n+i}|^2 = 1$.

It remains to identify the integer numbers n and i, Eq.(34), which achieve the lowest overlap, Eq.(30), for a given value μ_G, Eq.(31). According to our search the states which achieve minimum overlap almost everywhere are of the form $\psi_n |n\rangle + \psi_{n+3} |n+3\rangle$. More specifically, one finds that in the segment

$$\frac{1}{2n+1} \leq \mu_G \leq \frac{1}{2n+3}, \tag{35}$$

bridging the number states $|n\rangle$ and $|n+1\rangle$ (see Fig.1b), the states which achieve minimum are

$$|\psi\rangle_a = \psi_n |n\rangle + \psi_{n+3} |n+3\rangle \tag{36}$$

$$\text{for } \frac{1}{2n+1} \leq \mu_G \leq r_n$$

and

$$|\psi\rangle_b = \psi_{n-2} |n-2\rangle + \psi_{n+1} |n+1\rangle \tag{37}$$

$$\text{for } r_n \leq \mu_G \leq \frac{1}{2n+3}$$

where the parameter r_n is the real root of the equation

$$x^4 - 2(1+n)x^3 - 4x^2 - 6(1+n)x + 3 = 0$$

in the interval $[\frac{1}{2n+1}, \frac{1}{2n+3}]$. We note here that such a root always exists though we do not present its form explicitly here.

The overlap that corresponds to the states $|\psi\rangle_a$ and $|\psi\rangle_b$ can be easily evaluated with the help of Eqs.(30)-(31),

$$\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right)_a = \frac{1}{3\left(1+\mu_G\right)}\left(\left(7\mu_G + 2n\mu_G - 1\right)\left(\frac{1-\mu_G}{1+\mu_G}\right)^n + \left(1-\mu_G - 2n\mu_G\right)\left(\frac{1-\mu_G}{1+\mu_G}\right)^{n+3}\right), \qquad (38)$$

and

$$\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right)_b = \frac{1}{3\left(1+\mu_G\right)}\left(\left(1+3\mu_G - 2n\mu_G\right)\left(\frac{1-\mu_G}{1+\mu_G}\right)^{n+1} + \left(-1+3\mu_G + 2n\mu_G\right)\left(\frac{1-\mu_G}{1+\mu_G}\right)^{n-2}\right). \qquad (39)$$

However the state $|\psi\rangle_b$, Eq.(37), is not defined when $n=0$ or 1. In this case the minimum $\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right)$ is rather obtained by the states

$$|\psi\rangle_\beta = \psi_n\,|n\rangle + \psi_{n+1}\,|n+1\rangle \qquad (40)$$

with $n = 0$, 1 respectively. The states in Eq.(40) do not possess a un-shifted thermal reference Gaussian state and, apart from numerical evidence, we do not have a formal proof for their extremal properties. Using the Wigner function for the states Eq.(40) expressed in terms of Laguerre polynomials as

$$\begin{aligned} W_n\left(x,p\right) = {} & \frac{\left(-1\right)^n}{\pi}\mathrm{e}^{-x^2-p^2} \\ & \times\Big(\left(\left|\psi_n\right|^2 L_n\left(2x^2+2p^2\right)\right. \\ & - \left|\psi_{n+1}\right|^2 L_{n+1}\left(2x^2+2p^2\right) \\ & \left. + \frac{2\sqrt{2}}{\sqrt{n+1}}\,\mathrm{Re}\left(\psi_n\psi^*_{n+1}\left(x-\mathrm{i}p\right)\right)L^1_n\left(2x^2+2p^2\right)\Big)\right). \end{aligned} \qquad (41)$$

we have calculated the corresponding overlaps

$$\begin{aligned} \mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right)_{\beta,0} = {} & -e^{\frac{(\alpha-1)\alpha}{2\alpha^2-3\alpha+2}} \\ & \times\frac{-2\alpha^5+8\alpha^4-12\alpha^3+5\alpha^2+4\alpha-4}{(2-\alpha)^{3/2}\left(2\alpha^2-3\alpha+2\right)^{5/2}}, \end{aligned} \qquad (42)$$

$$\begin{aligned} \mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right)_{\beta,1} = {} & -\frac{2e^{\frac{2(\alpha-1)\alpha}{4\alpha^2-5\alpha+3}}}{(3-\alpha)^{5/2}\left(4\alpha^2-5\alpha+3\right)^{9/2}} \\ & \times\left(64\alpha^{10} - 560\alpha^9 + 2156\alpha^8 - 4668\alpha^7\right. \\ & + 6004\alpha^6 - 4211\alpha^5 + 494\alpha^4 + 1938\alpha^3 \\ & \left. - 1908\alpha^2 + 837\alpha - 162\right). \end{aligned} \qquad (43)$$

The parameter $\alpha = \left|\psi_n\right|^2 = 1 - \left|\psi_{n+1}\right|^2$ is defined by the corresponding equations for μ_G

$$\mu_{G_{\beta,0}} = -1/\sqrt{2\alpha-3}\sqrt{4(1-\alpha)\alpha+2\alpha-3} \qquad (44)$$

$$\mu_{G_{\beta,1}} = -1/\sqrt{2\alpha-5}\sqrt{8(1-\alpha)\alpha+2\alpha-5} \qquad (45)$$

We unify the results, Eqs.(38)-(39), (42)-(45), and we present them graphically in Fig. **??**. This graph simply represents the minimum value of the uncertainty $\sqrt{\gamma_{11}\gamma_{22} - \left|\gamma_{12}\right|^2} = 1/\mu_G$ of a pure state given the value of its overlap $\mathrm{Tr}\left(\hat{\rho}\hat{\rho}_G\right)$ (or the non-Gaussianity δ). The line in the inset formally stands for the function $F\left(\mu=1,\delta\right)$

$$\sqrt{\gamma_{11}\gamma_{22} - \left|\gamma_{12}\right|^2} \geqslant F\left(1,\delta\right)$$

(see Eq.(8)), which we have not explicitely derive since this requires an inversion of Eqs.(38)-(39).*Interestingly this extended version of the Schrödinger-Robertson uncertainty relation is saturated not only by the ground state of the harmonic oscillator but by all number states.*

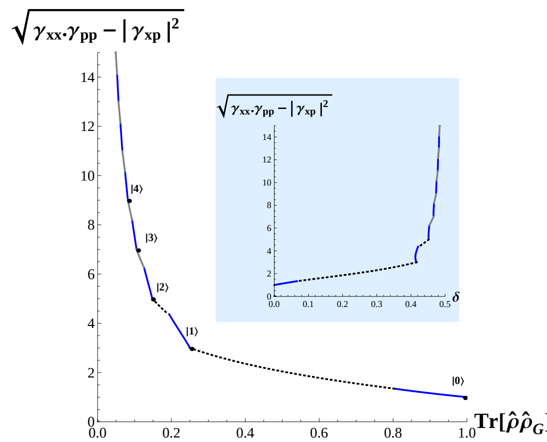

FIG. 2: An extended version of the Scrödinger-Robertson relation for *pure* states Eq.(III B 1) where the minimum value on the uncertainty $\sqrt{\gamma_{11}\gamma_{22} - \left|\gamma_{12}\right|^2}$ depends on the quantity $\mathrm{Tr}(\hat{\rho}_G\hat{\rho})$ (or δ). The set of minimizing states is comprised by the states $|\psi\rangle_a$ Eq.(38) *solid blue line*, $|\psi\rangle_b$ Eq.(39) *solid gray line*, and $|\psi\rangle_\beta$ Eqs.(42)-(45) *dotted line*. The number states are included in the set of minimizing states and are marked on the figure as dots.

2. *Mixed states*

Moving away from the plane of pure states (density matrices of rank 1) it is logical to proceed with the search for extremal states starting from density matrices of low rank. The first case to be considered is that of *rank* 2. According to the spectral theorem, a unique pair of mutually orthogonal pure states $|\psi_1\rangle$, $|\psi_2\rangle$ always exists such that $\hat{\rho} = p_1\,|\psi_1\rangle\langle\psi_1| + p_2\,|\psi_2\rangle\langle\psi_2|$. We consider

now that the probability mixing coefficients p_1 and p_2 are fixed and we look for the states $|\psi_i\rangle$ for which the overlap becomes minimum. As before we assume that $\hat{\rho}$ possess a reference Gaussian state with no angular dependence on the phase space and we impose the constrains of normalization $\langle\psi_i|\psi_i\rangle = 1$ and fixed covariance matrix Eq.(28). As we did in the special case of pure states, we decompose the states $|\psi_i\rangle$ in the eigenbasis of the harmonic oscillator $|\psi_i\rangle = \sum_n \psi_{i,n}|n\rangle$ and apply the method of Lagrange multipliers by differentiating over the amplitudes $\psi_{i,n}$. The conditions on the vectors $|\psi_i\rangle$ that we obtain are the following

$$\left(e^{\beta\hat{n}} + c_1 + c_2\hat{n}\right)|\psi_1\rangle = 0, \tag{46}$$
$$\left(e^{\beta\hat{n}} + c_3 + c_2\hat{n}\right)|\psi_2\rangle = 0. \tag{47}$$

From the Eqs.(46)-(47) we conclude that in addition to simple mixtures of number states ($|\psi_1\rangle = |i\rangle$ and $|\psi_2\rangle = |j\rangle$) which we have discussed already in Sec. IIIA, there is an additional solution of the form

$$|\psi_1\rangle = \psi_{1,i}|i\rangle + \psi_{1,j}|j\rangle \tag{48}$$
$$|\psi_2\rangle = |k\rangle \tag{49}$$

where $j \geqslant i+3$ and $k \neq i, j$. With simple observation we have arrived to the conclusion that most probably among the *rank* 2 states of this form the ones which achieve the minimum overlap (with some exception that we discuss below) are the states, Eqs.(48)-(49), where $j = i+3$ and $k = i+1$ or $i+2$. More precisely, the states of the minimum overlap are

$$\begin{aligned}\hat{\rho}_i = p&\left((\psi_n|n\rangle + \psi_{n+3}|n+3\rangle)\right)\left(\psi_n^*\langle n| + \psi_{n+3}^*\langle n+3|)\right)\\ &+ (1-p)|n+i\rangle\langle n+i|\end{aligned} \tag{50}$$

where $i = 1$ or 2 and $|\psi_n|^2 + |\psi_{n+3}|^2 = 1$. When $p = 0$, Eq.(50), the solution Eq.(36) for pure states is recovered, while for $\psi_n = 0$ and $i = 2$ (or $\psi_{n+3} = 0$ and $i = 1$) one arrives to the rank 2 states Eq.(26) of the *Region I*. As for pure states, for mixed states the *Region II* of the bound is not covered totally by the states suggested by the optimization problem but there is part that the (shifted) states

$$\begin{aligned}\hat{\rho}_3 = &a|n\rangle\langle n| + (1-a)|n+1\rangle\langle n+1|\\ &+ b|n\rangle\langle n+1| + b^*|n+1\rangle\langle n|,\end{aligned} \tag{51}$$

seem to intervene, where $|b| \in \left[-\sqrt{a(1-a)}, \sqrt{a(1-a)}\right]$ and $n = 0, 1$. In the limiting case where $|b| = \sqrt{a(1-a)}$ the corresponding states $|\psi\rangle_\beta$ Eq. (40) are recovered from the Eq.(51).

For the states $\hat{\rho}_1$, $\hat{\rho}_2$ the quantities of interest, $\mathrm{Tr}(\hat{\rho}_G\hat{\rho})$ and μ_G, are given by Eqs.(30)-(31). For the states $\hat{\rho}_3$ the calculations are more involved, and for this reason we just present the results below,

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}_G\hat{\rho})_{3,0} = &e^{\frac{|b|^2}{a+2|b|^2-2}}\\ &\times\frac{\left(2(a-1)|b|^4 + 2(a-2)a|b|^2 + (a-2)^2\right)}{(2-a)^{3/2}\left(-a-2|b|^2+2\right)^{5/2}}\end{aligned} \tag{52}$$

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}_G\hat{\rho})_{3,1} = &-\frac{e^{\frac{2|b|^2}{a+4|b|^2-3}}}{(3-a)^{5/2}\left(-a-4|b|^2+3\right)^{9/2}}\\ &\times 2\left(16\left(4a^2-15a+15\right)|b|^8\right.\\ &+ 2(a-3)^2\left(10a^2-9a-25\right)|b|^4\\ &+ 2(a-3)^3\left(a^2+3a-10\right)|b|^2\\ &+ (a-3)^4(a-2)\\ &\left.+8\left(8a^3-45a^2+70a-21\right)|b|^6\right)\end{aligned} \tag{53}$$

where the parameters a and b are the same as in Eq.(51). The latter connect indirectly (in a parametric way) the overlap with the other quantities of interest,

$$\mu = a^2 + (1-a)^2 + 2|b|^2 \tag{54}$$

$$\mu_{G_{3,n}} = 1/\sqrt{(2a-2n-3)\left(2a+4|b|^2(n+1)-2n-3\right)} \tag{55}$$

with $n = 0$ or 1.

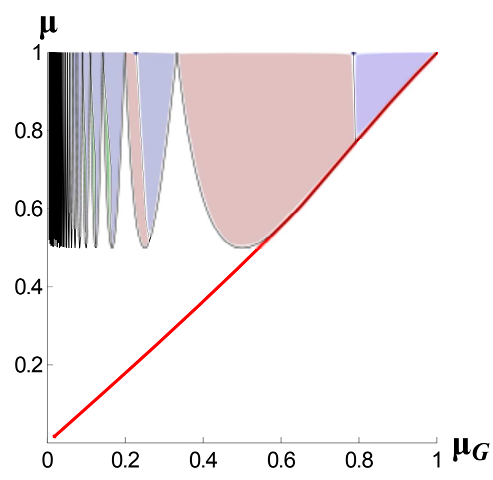

FIG. 3: The *region II* of the bound projected on the plane $\{\mu, \mu_G\}$. We attribute to the bound three different colors; *Blue* for the part realized by the states $\hat{\rho}_0$ Eq.(50), *Green* for the satets $\hat{\rho}_1$ Eq.(50), and *Red* for the states $\hat{\rho}_3$ Eq.(51).

If the analogous extremization procedure is applied to mixed states of higher rank, then the number of the obtained conditions on the solution is increasing linearly with the rank. For instance, for the rank 3 states which

can always be expressed as $\hat{\rho} = p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2| + p_3 |\psi_3\rangle\langle\psi_3|$ with $\langle\psi^i|\psi^j\rangle = \delta_{ij}$ and $\sum p_i = 1$, one obtains in addition to the conditions Eqs.(46)-(47), the condition

$$\left(e^{-\beta n} + c_4 + c_2 n\right)|\psi_3\rangle = 0.$$

This suggests an extreme solution of a form, e.g., $\hat{\rho}_i + |m\rangle\langle m|$, where $\hat{\rho}_i$ is defined in Eq.(50) and $|m\rangle$ a Fock state with $m \neq n$, $n+3$, $n+i$. After investigation of different possibilities, we saw that such states do not achieve overlap lower than the *rank* 2 ones. This fact led us to the conclusion that the identified *rank* 2 density matrices, Eqs.(50)-(51) are the extreme solutions which realize the *Region II* of the bound.

In Fig. III B 2, in analogy with the Fig.1 (b), we present how the *Region II* of the bound, projected on the plane $\{\mu, \mu_G\}$, is separated to three different parts, realized by the states $\hat{\rho}_1$, $\hat{\rho}_2$ Eq.(50) and $\hat{\rho}_3$ Eq.(51) respectively.

C. The total bound

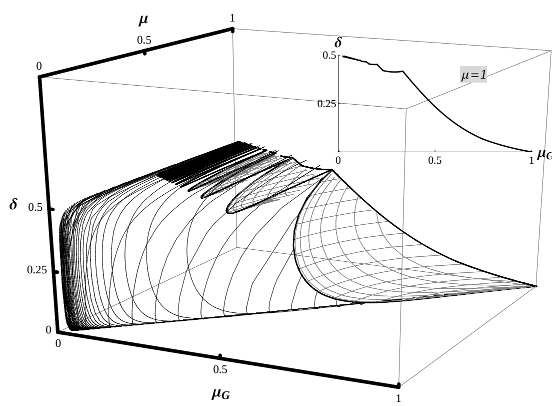

FIG. 4: The non-Gaussianity uncertainty relation as a bound in the $\{\mu_G, \mu, \delta\}$ parametric space. The part of the bound that is to the right of the curly black line represents the *Region II* of the bound and the rest the *Region I*. In the *inset* the intersection of the bound with the plane $\mu = 1$ is presented.

The states which minimize the purity (*Region I*) together with the states which maximize the purity (*Region II*) form a surface which bounds from below all the mixed states in the parametric space $\{\mu, \mu_G, \mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)\}$ or equivalently bounds from above in the space $\{\mu, \mu_G, \delta\}$. In Fig. 4 we present the total bound that we name *the non-Gaussianity bounded uncertainty relation.* Clearly the minimal uncertainty, $F(\mu, \delta)$, depends on the degree of non-Gaussianity of a state. Although the dependence is "smooth" for the low-purity region (up to purity 1/2) the structure of the bound becomes more complex for higher values of purity and for some very small regions the bound projected on the plane $\{\mu, \delta\}$ is double valued. For these areas clearly the function $F(\mu, \delta)$ in Eq.(8) cannot be defined. This problem does not appear when one uses the quantity $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ instead.

IV. THE HUDSON'S THEOREM AND UNCERTAINTY RELATION FOR MIXED STATES

According to the arguments in [7] the Hudson's theorem for mixed states can be formulated as an upper bound on the non-Gaussianity for the states with a positive Wigner function. The idea behind this formulation is that since among the pure states only the Gaussian states can possess a positive Wigner function, for mixed states one expects that the maximum degree of non-Gaussianity of a state with a positive Wigner function is lower than that of a generic state. In practice, this bound can be utilized as a set of experimentally verifiable, necessary conditions for a state to have a positive Wigner function or alternatively, as a set of sufficient conditions for a state to posses a non-positive Wigner function. Such conditions can be essential in the construction of "non-classical" states which are useful resources in protocols of quantum information [14].

Even though it is not trivial to identify the whole bound in the space $\{\mu, \mu_G, \delta\}$ for states with non-negative Wigner functions, we can conclude using the same arguments as in Sec. IIIA that the purity of the states with non-negative Wigner function for given μ_G and $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ is minimized by mixtures of number states. To our knowledge there is no necessary and sufficient condition that one can impose on the coefficients of a mixture of number states to guarantee positivity of the Wigner function. Due to this difficulty we cannot proceed with further analytical identification of the states which realize the Hudson's theorem for mixed states.

On the other hand from the results in Sec. III we can numerically obtain some information about the Hudson's theorem for mixed states. To do so, in the bound Fig.4 for all states we identify numerically the part realized by states with positive Wigner function. We depict this common area, where the bounds for all states and for states with strictly positive Wigner function are overlapping, as a shaded region in Fig.5. In [15] we have gone one step further and we have identified numerically the whole *region I* of the bound for the states with positive Wigner function. We have concluded that the two bounds stay very close when the purity of the state is low (for a fixed uncertainty) and we obtained an indication that the distance between the two bounds is increasing with purity. In addition in [15] the lowest degree of $\mathrm{Tr}\,(\hat{\rho}\hat{\rho}_G)$ that a state with strictly positive Wigner function may achieve for a given uncertainty, has been identified.

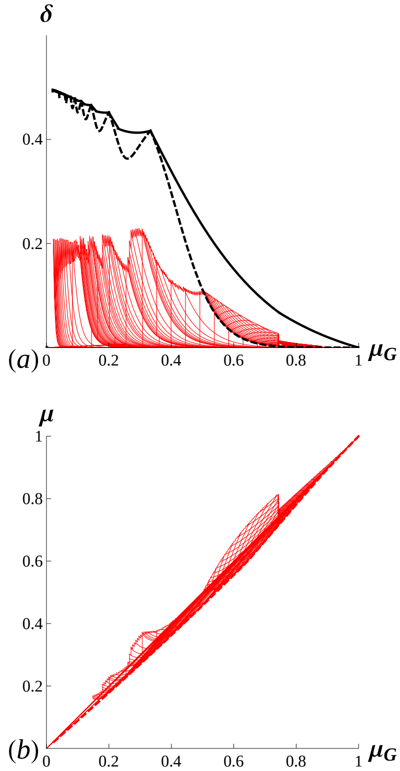

FIG. 5: The part of the bound (red) realized by states with positive Wigner function projected on the planes *(a)* $\{\mu_G, \delta\}$ and *(b)* $\{\mu_G, \mu\}$. In figure *(a)* it is plotted together with the upper non-Gaussianity bounds for all mixed states (solid line) achieved by pure states and the boundary line (dashed line) among the *Region I* and *II* realized by the rank 2 states Eq.(26).

V. CONCLUSIONS

Inspired by the discussion in [4] and some recent results on the extension of Hudson's theorem for mixed states [7], we have extended the purity-bounded uncertainty relation, adding one more parameter, namely, the degree of non-Gaussianity. According to the results presented here, the minimal uncertainty for quantum states strongly depends not only on the degree of mixedeness but also on its non-Gaussian character. The set of states which saturate this uncertainty relation includes the Gaussian states. From this we can conclude that even though Gaussianity is not a requirement for the minimization of this uncertainty relation, Gaussian states are still included in the set of minimizing states and they dominate as purity tends to one. Interestingly, the non-Gaussianity bounded uncertainty relation is valid for pure states and there it is saturated, among others, by all eigenstates of the harmonic oscillator which appear as extremal points of the bound.

Moreover, by identifying the states which possess a positive Wigner function among the ones which saturate the non-Gaussianity bounded uncertainty relation, we come closer to the extension of Hudson's theorem for mixed states. Gaussian mixed states (thermal states) both saturate the non-Gaussianity uncertainty relation and satisfy the extended Hudson's theorem. We should note that in [15] we employ numerical methods to compliment the results presented here concerning the Hudson's theorem. The main result in [15] is the identification of the minimum degree of trace overlap that a mixed state of fixed uncertainty may achieve.

In this work we have chosen purity as a measure of the degree of mixedeness of the state. If the von Neumann entropy is employed instead [8], then the results are similar to the one presented in this work and one arrives to the non-Gaussianity extension of the "entropy-bounded" uncertainty relation suggested by Bastiaans [6]. The advantage in using the entropy, is that the solution is automatically a positive matrix and one does not need to employ the ansatz we have used in the current work, to impose positivity (Eq.(19)). Furthermore, the solution to the extremization problem contains two "branches", one bounding the trace overlap from above and one from below for fixed uncertainty and purity. Note than in the current work we derive only the lower bound on the quantity of trace overlap. On the other hand, the states which minimize the extended uncertainty relation in [8] are mixtures of all number states with $n \to \infty$, the solution cannot be expressed in a closed form and one needs the help of numerical methods to visualize the results.

Acknowledgments

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Appendix A: A parametric expression for the purity-bounded uncertainty relation

Here, we derive an expression for the purity-bounded uncertainty relation alternative to the one presented in Ref. [5]. The derivation of the non-Gaussianity bounded uncertainty relation and its approximate formula in Sec. III, is a direct generalization to higher dimensions of the procedure we present here.

In [5] it is proven that states with angular-independent Wigner function minimize the relation. We use this result and we start with states which are finite convex combinations of the number states

$$\hat{\rho} = \sum_{n=0}^{N} P_n |n\rangle\langle n| \tag{56}$$

where

$$\sum_{n=0}^{N} P_n = 1 \tag{57}$$

and $0 \leq P_n \leq 1$. The purity of $\hat{\rho}$, and the purity of the reference Gaussian state are determined by

$$\sum_{n=0}^{N} P_n^2 = \mu\,, \tag{58}$$

$$\sum_{n=0}^{N} P_n(n+1/2) = 1/2\mu_G\,, \tag{59}$$

respectively. As in [4] we use the Lagrange multipliers method and we arrive to the following form for the coefficients of the states which minimize the purity $\mu[\hat{\rho}]$,

$$P_n^{ex} = A_1 + A_2 n \tag{60}$$

where $A_1$and A_2 are to be determined by the conditions Eqs.(57) and (59). However the parameter N, the upper limit in the summation, remains undetermined and to resolve this ambiguity we introduce a continuous parameter y that satisfies

$$A_1 + A_2 y = 0. \tag{61}$$

This additional equation permits us to define N as the integer part of y, $N = \lfloor y \rfloor$.

We employ the conditions, Eqs. (57) and (61) , and express $A_1$and A_2 in terms of N and y. Then we substitute the result into the Eqs.(58),(59) and we obtain a parametric relation for the extremal solution provided by the Lagrange multiplier's method,

$$\mu_G^{ex} = \frac{3\,(N-2y)}{N\,(5+4N) - 6\,(1+N)\,y}, \tag{62}$$

$$\mu^{ex} = \frac{2\left(N+2N^2-6Ny+6y^2\right)}{3\,(1+N)\,(N-2y)^2}\,. \tag{63}$$

where $N = \lfloor y \rfloor$ and $y \in [1,\infty)$. We note that by setting $N = y$ one can arrive to an approximate formula

$$\mu^{ex} = \frac{8\mu_G^{ex}}{9 - \left(\mu_G^{ex}\right)^2}$$

that reproduces the curve of Eqs.(62)-(63) in very good approximation when $0 \leq \mu_G \leq 3/5$. For the rest of the region, $3/5 \leq \mu_G \leq 1$, one should better employ the exact solution

$$\mu_{N=1} = \frac{1-4\mu_G+5\mu_G^2}{2\mu_G^2}. \tag{64}$$

The obtained results Eqs.(62)-(63) give the lower bound (see Fig.1(a)) on the plane μ_G and μ and are equivalent to the expressions derived in [5].

Appendix B: Parametric expression for the non-Gaussianity Bounded uncertainty relation

Here we present the explicit expressions for the trace overlap and the purity that realize the part of the non-Gaussianity bounded uncertainty relation covered by mixtures of number states.

$$\begin{aligned} Tr\,(\rho_G\rho)^{ex} =& \frac{1}{r}\left(\frac{1-r}{r+1}\right)^{\mathrm{N_{min}}}\left(2A_1 r + 2A_2\mathrm{n_{min}}r - A_2 r + A_2 + A_3 r^2\left(\frac{1-r}{r+1}\right)^{\mathrm{n_{min}}}\right) + \\ & \frac{1}{r}\frac{(1-r)}{(r+1)^2}\left(\frac{1-r}{r+1}\right)^{\lfloor x\rfloor}\left(r\left(A_3(r-1)r\left(\frac{1-r}{r+1}\right)^{\lfloor x\rfloor} - 2A_1(r+1)\right) - A_2(r+1)(2\lfloor x\rfloor\, r + r + 1)\right) \end{aligned} \tag{65}$$

$$\begin{aligned} \mu^{ex} =& -\frac{1}{6}(\mathrm{n_{min}} - \lfloor x\rfloor - 1)\left(6{A_1}^2 + 6A_1A_2(\mathrm{n_{min}} + \lfloor x\rfloor) + {A_2}^2\left(2\mathrm{n_{min}}\lfloor x\rfloor + \mathrm{n_{min}}(2\mathrm{n_{min}}-1) + 2\lfloor x\rfloor^2 + \lfloor x\rfloor\right)\right) \\ & + A_3\frac{1}{4r}\left(\frac{1-r}{r+1}\right)^{\mathrm{n_{min}}}\left(r\left(4A_1 + \mathrm{A}_3 r\left(\frac{1-r}{r+1}\right)^{\mathrm{n_{min}}}\right) + A_2((4\mathrm{n_{min}}-2)r+2)\right) \\ & - A_3\frac{(r-1)}{4r(r+1)^2}\left(\frac{1-r}{r+1}\right)^{\lfloor x\rfloor}\left(r\left(A_3(r-1)r\left(\frac{1-r}{r+1}\right)^{\lfloor x\rfloor} - 4\mathrm{A}_1(r+1)\right) - 2A_2(r+1)(2\lfloor x\rfloor\, r + \mu + 1)\right) \end{aligned} \tag{66}$$

where r stands for μ_G, x for x_2 and A_1, A_2, and A_3 are defined by the linear system of equations

$$A_1 + A_2 x + A_3 r \left(\frac{1-r}{r+1}\right)^x / (r+1) = 0 \tag{67}$$

$$A_1(-\mathrm{n}_{\min} + \lfloor x \rfloor + 1) + A_2((1-\mathrm{n}_{\min})\mathrm{n}_{\min} + \lfloor x \rfloor(\lfloor x \rfloor + 1))/2 + A_3 \left(\left(\frac{1-r}{r+1}\right)^{\mathrm{n}_{\min}} - \left(\frac{1-r}{r+1}\right)^{\lfloor x \rfloor + 1} \right)/2 = 1 \tag{68}$$

$$\begin{aligned} &A_1 \left(-\mathrm{n}_{\min}{}^2 + \lfloor x \rfloor^2 + 2\lfloor x \rfloor + 1\right) + \frac{1}{6} A_2 \left(-4\mathrm{n}_{\min}{}^3 + 3\mathrm{n}_{\min}{}^2 + \mathrm{n}_{\min} + \lfloor x \rfloor \left(4\lfloor x \rfloor^2 + 9\lfloor x \rfloor + 5\right)\right) \\ &+ A_3 \left(\left(\frac{1-r}{r+1}\right)^{\mathrm{n}_{\min}} (2\mathrm{n}_{\min} r + 1) - \left(\frac{1-r}{r+1}\right)^{\lfloor x \rfloor + 1} (2(\lfloor x \rfloor + 1) r + 1) \right)/2r = 1/r. \end{aligned} \tag{69}$$

One should first solve the above system of linear equations to express A's in terms of the parameters r, x and $\mathrm{n}_{\min}$ and substitute the resulting expressions in Eqs.(65)-(66). One should then attribute an integer non-negative value to $n_{\min}$. In this way the running parameters left are $r \in [0,1]$ and $x \in [2,\infty]$. However their limits are furhter constrained by

$$A_1 + A_2 n_{\min} + A_3 r \left(\frac{1-r}{r+1}\right)^{n_{\min}} / (r+1) > 0, \quad \forall\, n_{\min} \tag{70}$$

$$A_1 + A_2 (n_{\min} - 1) + A_3 r \left(\frac{1-r}{r+1}\right)^{(n_{\min}-1)} / (r+1) < 0, \quad \textit{if}\ n_{\min} \neq 0 \tag{71}$$

which ensure that $n_{\min}$ remains constant.

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